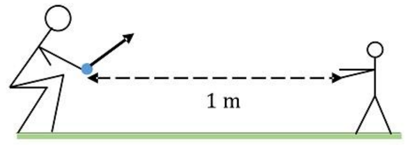
PHYS 101 – General Physics I Final Exam

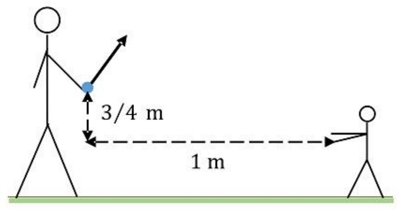
Duration: 120 minutes　　　　Sunday, 24 December 2023; 15:00

1. Small children have difficulty catching objects which are moving fast, yet they enjoy playing catch. So any adult playing with a small child tries to throw the ball so that it reaches the child with the **smallest possible speed.**

(a) (12 Pts.) Assume that you crouch to the level of the child so that the throwing point and catching point are at the same height. What are the x and y components of the initial velocity of the ball so that it reaches a child 1 m away with the least possible kinetic energy? (Use $g = 10\ \mathrm{m/s^2}$ and ignore air resistance.)



(b) (13 Pts.) Assume that you are throwing the ball from a point $3/4$ m higher than the child who is 1 m away horizontally. What are the x and y components of the initial velocity of the ball so that it reaches the child with minimum kinetic energy? (Use $g = 10\ \mathrm{m/s^2}$ and ignore air resistance.)



Solution:

Kinematical equations describing projectile motion, and the expression for kinetic energy are

$$x = x_0 + v_{0x}t\,, \qquad y = y_0 + v_{0y}t - \frac{1}{2}g\,t^2\,, \qquad K = \frac{1}{2}m\left(v_{0x}^2 + v_{0y}^2\right).$$

To minimize the kinetic energy, we need to find the relation between v_{0x} and v_{0y}.

(a) Taking the origin at the initial position of the ball, we have

$$x = v_{0x}t\,, \qquad y = v_{0y}t - 5\,t^2.$$

The ball is caught by the child at $x = 1$ m and $y = 0$. So

$$1 = v_{0x}t \;\rightarrow\; t = \frac{1}{v_{0x}}\,, \qquad v_{0y}t - 5\,t^2 = 0 \;\rightarrow\; v_{0y} = \frac{5}{v_{0x}} \;\rightarrow\; K = \frac{1}{2}m(v_{0x}^2 + 25\,v_{0x}^{-2})\,.$$

Since in this case final kinetic energy is equal to initial kinetic energy, its minimum value requires

$$\frac{dK_i}{dv_{0x}} = m(v_{0x} - 25\,v_{0x}^{-3}) = 0 \;\rightarrow\; v_{0x}^4 = 25 \;\rightarrow\; v_{0x} = \sqrt{5}\ \mathrm{m/s}\,, \qquad v_{0y} = \frac{5}{v_{0x}} = \sqrt{5}\,.$$

(b) With the same choice of the origin, we now have

$$x = v_{0x}t\,, \qquad y = \frac{3}{4} + v_{0y}t - 5\,t^2.$$

Again, the ball is caught by the child at $x = 1$ m and $y = 0$. So

$$1 = v_{0x}t \;\rightarrow\; t = \frac{1}{v_{0x}}\,, \qquad \frac{3}{4} + v_{0y}t - 5\,t^2 = 0 \;\rightarrow\; v_{0y} = -\frac{3}{4}v_{0x} + \frac{5}{v_{0x}}.$$

$$K_i = \frac{1}{2}m\left(v_{0x}^2 + \frac{9}{16}v_{0x}^2 + 25\,v_{0x}^{-2} - \frac{15}{2}\right) = \frac{1}{2}m\left(\frac{25}{16}v_{0x}^2 + 25\,v_{0x}^{-2} - \frac{15}{2}\right).$$

Since $K_f = K_i + mgh$, where mgh is a constant, we have

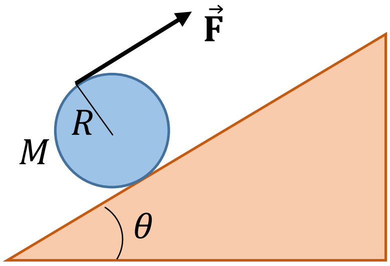
$$\frac{dK_f}{dv_{0x}} = \frac{dK_i}{dv_{0x}} = 25\,m\left(\frac{v_{0x}}{16} - v_{0x}^{-3}\right) = 0 \;\rightarrow\; v_{0x}^4 = 16 \;\rightarrow\; v_{0x} = 2\ \mathrm{m/s}\,, \qquad v_{0y} = -\frac{3}{2} + \frac{5}{2} = 1\ \mathrm{m/s}\,.$$

2. A cylinder of mass M, radius R, and rotational inertia $I = MR^2/2$ is placed on an inclined plane whose angle of inclination is θ. A string is wound around the cylinder and pulled up with a force $\vec{\mathbf{F}}$ parallel to the incline. The coefficient of friction is large enough to prevent slipping of the cylinder on the inclined plane, and the string does not slip on the cylinder. Gravitational acceleration is g.

(a) (5 Pts.) What is the magnitude F of the force needed to keep the cylinder in equilibrium?

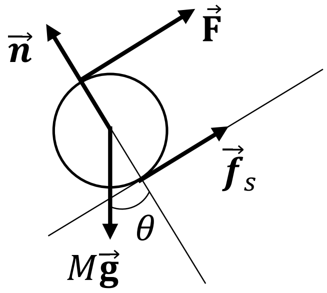
(b) (10 Pts.) Find the acceleration if F is large enough so that the cylinder accelerates up the inclined plane without slipping.

(c) (10 Pts.) What is the minimum value of the coefficient of static friction for the cylinder not to slip when it is moving up under the action of the force $\vec{\mathbf{F}}$?



Solution: (a) We need to have the net torque equal to zero for equilibrium. Evaluating torques with respect to the point of contact, we have

$$2R\,F - MgR\sin(\pi - \theta) = 0 \quad \rightarrow \quad F = \frac{1}{2}Mg\sin\theta$$



(b) If the cylinder accelerates up the inclined plane without slipping, we have $a = R\alpha$, where a is the linear acceleration of the center, and α is the angular acceleration of the cylinder about its symmetry axis. Newton's scond law is written as

$$n - Mg\cos\theta = 0\,, \qquad F + f_s - Mg\sin\theta = Ma\,, \qquad RF - Rf_s = I\,\alpha \;\rightarrow\; F - f_s = \frac{1}{2}\,Ma\,.$$

$$a = \frac{4F}{3M} - \frac{2}{3}g\sin\theta\,.$$

(c) Solving the above equations for f_s, we find

$$f_s = \frac{1}{3}F + \frac{1}{3}Mg\sin\theta\,.$$

Since $f_s \leq \mu\,n$ and $n = Mg\cos\theta$, we find

$$\frac{1}{3}F + \frac{1}{3}Mg\sin\theta \leq \mu Mg\cos\theta \quad \rightarrow \quad \mu \geq \frac{F + Mg\sin\theta}{3Mg\cos\theta}\,.$$

3. Two planets are observed to have circular orbits around a far away star. The first planet has mass m_1 and completes its orbit of radius R_1 in time T_1. The second planet's orbital radius is R_2. The star is much more massive than the planets, so the gravitational interaction between the planets can be neglected.

(a) (7 Pts.) What is the mass of the star in terms of given quantities and universal constants?

(b) (7 Pts.) What is the orbital period of the second planet?

(c) (7 Pts.) What is the ratio of the orbital speeds of the two planets in terms of R_2/R_1?

(d) (4 Pts.) Can you determine the mass of the second planet with the given information? Why/Why not?

Solution:

(a) Let M denote the mass of the star, and m_1, m_2 denote masss of the planets. Newton's second law $F = ma$ applied to circular orbits implies

$$\frac{GMm}{r^2} = m\frac{v^2}{r} \quad \rightarrow \quad v_1 = \sqrt{\frac{GM}{R_1}}, \qquad v_2 = \sqrt{\frac{GM}{R_2}}.$$

Since

$$T = \frac{2\pi r}{v} \quad \rightarrow \quad T_1 = \frac{2\pi R_1}{v_1} = 2\pi\sqrt{\frac{R_1^3}{GM}} \quad \rightarrow \quad T_1^2 = \left(\frac{4\pi^2}{GM}\right)R_1^3 \quad \rightarrow \quad M = \left(\frac{4\pi^2}{G}\right)\left(\frac{R_1^3}{T_1^2}\right).$$

(b) Similarly,

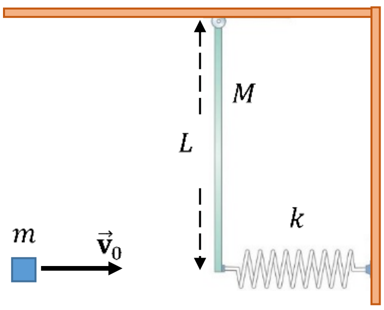
$$T_2^2 = \left(\frac{4\pi^2}{GM}\right)R_2^3 \quad \rightarrow \quad \frac{T_1^2}{R_1^3} = \frac{T_2^2}{R_2^3} = \left(\frac{4\pi^2}{GM}\right) \quad \rightarrow \quad T_2 = T_1\left(\frac{R_2}{R_1}\right)^{3/2}.$$

(c)

$$\frac{v_1}{v_2} = \sqrt{\frac{R_2}{R_1}}$$

(d) No, we can not. Expressions for the speed and the period are independent of the mass of the planets. So, a planet with any mass traces the same orbit. (Equivalence of the gravitational and inertial mass.)

4. A uniform rod of length L and mass M ($I_{CM} = ML^2/12$) is at rest on a frictionless horizontal surface. One end of the rod is pivoted on a frictionless hinge fixed to a wall, while the other end is fixed to a spring with stiffness constant k. The other end of the spring is fixed to a wall. A block of mass m sliding across the frictionless horizontal surface with speed v_0 perpendicular to the rod makes a completely inelastic collision with the rod, sticking to the end of the rod after the collision. Assume that the collision is instantaneous, and that the compression of the spring from its equilibrium length after the collision is small. (The figure illustrates the motion as seen from above.)



(a) (12 Pts.) What is the maximum compression of the spring?

(b) (13 Pts.) What is the period of small oscillations of the system?

Solution:

(a) Angular momentum with respect to the pivot is conserved in the collision. This can be used to find the angular speed of the rod plus the block immediately after the collision.

$$L_i = mv_0L\,, \qquad L_f = I\omega\,, \qquad I = \frac{1}{3}ML^2 + mL^2 = \frac{1}{3}(M+3m)L^2 \quad \rightarrow \quad \omega = \frac{3mv_0}{(M+3m)L}$$

Following the collision total mechanical energy is conserved. Hence the kinetic energy right after the collision is equal to the potential energy stored in the spring when the motion stops at maximum compression x_m.

$$\frac{1}{2}I\omega^2 = \frac{1}{2}kx_m^2 \quad \rightarrow \quad x_m^2 = \frac{I}{k}\omega^2 = \frac{(M+3m)L^2}{3k}\,\frac{9m^2v_0^2}{(M+3m)^2L^2} = \frac{3m^2v_0^2}{k(M+3m)} \quad \rightarrow \quad x_m = \sqrt{\frac{3m^2v_0^2}{k(M+3m)}}\,.$$

(b) For small compression x of the spring, the spring is approximately horizontal, and the restoring torque is

$$\tau = -kxL = -kL^2\sin\theta\,, \qquad \tau = I\alpha \quad \rightarrow \quad I\frac{d^2\theta}{dt^2} = -kL^2\sin\theta\,.$$

Since $\sin\theta \approx \theta$ for small oscillations, we have

$$I\frac{d^2\theta}{dt^2} + kL^2\theta = 0 \quad \rightarrow \quad \frac{d^2\theta}{dt^2} + \frac{3k}{M+3m}\theta = 0 \quad \rightarrow \quad \omega_0 = \sqrt{\frac{3k}{M+3m}} \quad \rightarrow \quad T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{M+3m}{3k}}\,.$$

Alternate solution: The expression for the total energy of the vibrating system is

$$E = \frac{1}{2}I\omega^2 + \frac{1}{2}kx^2 = \frac{1}{2}I\omega^2 + \frac{1}{2}kL^2\sin^2\theta.$$

Since energy is constant, we have

$$\frac{dE}{dt} = I\omega\frac{d\omega}{dt} + kL^2\sin\theta\cos\theta\frac{d\theta}{dt} = 0 \quad \rightarrow \quad I\frac{d^2\theta}{dt^2} + kL^2\sin\theta\cos\theta = 0\,.$$

Using $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ for small angles, we get the same equation.